

# Lattice Monte Carlo Data versus Perturbation Theory.

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Differences between lattice Monte Carlo data and perturbation theory (for example the lack of asymptotic scaling) are usually associated with the ‘bad’ behaviour of the bare lattice coupling  $g_0$  due to the effects of large (and unknown) higher order terms in  $g_0$ . In this philosophy a new, renormalised coupling  $g'$  is defined with the aim of making the higher order coefficients of the perturbative series in  $g'$  as small as possible.

In this paper an alternative scenario is discussed where lattice artifacts are proposed as the cause of the disagreement between Monte Carlo data and the  $g_0$ -perturbative series. We find that with the addition of a lattice artifact term, the usual asymptotic scaling expression in  $g_0$  is in excellent agreement with Monte Carlo data. Lattice data studied includes the string tension, the hadronic scale  $r_0$ , the discrete beta function,  $M_\rho$ ,  $f_\pi$  and the 1P-1S splitting in charmonium.

## 1. Introduction

A necessary condition for lattice predictions of QCD and other asymptotically free theories to have physical (continuum) relevance is that they reproduce weak coupling perturbation theory (PT) in the limit of the bare coupling  $g_0 \rightarrow 0$ . This perturbative scaling behaviour (a.k.a. asymptotic scaling) has not yet been observed for complicated theories like QCD when the *bare* lattice coupling  $g_0$  is used as the expansion parameter.

As a result of this disappointing disagreement, various workers have proposed methods of improving the convergence of the perturbation series by a re-expansion in terms of some new coupling  $g'$  [1,2].

This paper studies an alternative viewpoint in which the disagreement stems from lattice artifacts [3]. In this talk, it is shown that these terms can provide the mismatch between the lattice Monte Carlo data and  $g_0$ -PT without resorting to the use of a re-defined coupling  $g'$ .

The QCD quantities studied in this analysis are: the string tension,  $\sqrt{\sigma}$ ; the hadronic scale,  $r_0$  [4];  $M_\rho$ ;  $f_\pi$ ; the 1P-1S splitting in charmonium; and the discrete beta function  $\Delta\beta$ .

The results discussed here are presented in greater detail in [5].

## 2. Lattice Distorted-Perturbation Theory

Two-loop perturbation theory predicts the running of the lattice spacing  $a$  with coupling  $g^2$  as follows,

$$a^{-1}(g^2) = \frac{\Lambda}{f_{PT}(g^2)}, \quad \text{where} \quad (1)$$

$$f_{PT}(g^2) = e^{-\frac{1}{2b_0 g^2}} (g^2)^{\frac{-b_1}{2b_0^2}}$$

Lattice calculations predict  $a$  by calculating some dimensionful quantity on the lattice, and comparing it with its experimental value. As is well known these lattice values do not follow the above perturbative behaviour (when the bare coupling  $g_0$  is used). There are a number of possible causes of the disagreement: quenching; finite volume effects; unphysically large value of the quark mass; a real non-perturbative effect; the inclusion of only a finite number of terms (i.e. two) in the PT expansion; and lattice artifacts due to the finiteness of  $a$ . For the reasons outlined in [3], the first three effects cannot give rise to the sizeable discrepancy between lattice data and PT. As far as true (i.e. continuum) non-perturbative effects are concerned, the overwhelming expectation is that for cut-offs of  $a^{-1} \gtrsim 2$  GeV these effects should be minimal. Therefore the disagreement can only be due to either or both of the last two possibilities.

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Fitting Method		$a^{-1}$ from					$\Delta\beta(\beta)$
		$\sqrt{\sigma}$	$r_0$	$M_\rho$	$f_\pi$	$1P-1S$	
<b><math>g_0</math>-PT</b>	$\Lambda$ [MeV]	1.254(3)	1.599(5)	1.63(1)	1.48(2)	1.45(5)	—
	$\chi^2/dof$	484	262	10	9	6	702
<b>Leading-Order</b>	$X_n^\alpha$	0.204(2)	0.150(2)	0.22(2)	0.34(3)	0.35(6)	0.373(5)
	$\Lambda$ [MeV]	1.90(1)	1.958(9)	2.15(5)	2.2(1)	2.5(3)	—
<b>LDPT</b>	$\chi^2/dof$	3	16	1.1	1.6	0.3	4.2
	$X_n^\alpha$	0.26(2)	0.29(1)	—	—	—	0.24(1)
<b>Next-to-Leading-Order</b>	$X_{n+2}^\alpha$	-0.024(6)	-0.046(3)	—	—	—	0.050(5)
	$\Lambda$ [MeV]	1.96(2)	2.14(2)	—	—	—	—
<b>LDPT</b>	$\chi^2/dof$	1.7	1.4	—	—	—	1.7
<b><math>g_{MS}</math>-PT</b>	$\Lambda$ [MeV]	17.34(4)	20.89(7)	21.4(2)	21.0(3)	19.3(7)	—
	$\chi^2/dof$	160	47	1.3	2.5	1.5	78
<b><math>g_E</math>-PT</b>	$\Lambda$ [MeV]	4.81(1)	5.56(2)	5.77(4)	5.58(7)	5.2(2)	—
	$\chi^2/dof$	52	15	3.6	1.4	0.3	19

In this section we study the effect of lattice artifacts. These can be parametrised (to leading order) by modifying eq.(1) as follows:

$$a_L^{-1}(g_0^2) = \frac{\Lambda}{f_{PT}(g_0^2)} \times \left[ 1 - X_n^\alpha \frac{g_0^\alpha f_{PT}^n(g_0^2)}{f_{PT}^n(1)} \right], \quad (2)$$

(with no implicit summation over  $\alpha$  or  $n$ ).

Note that the  $\mathcal{O}(a^n)$  coefficient in Eq(2) has been normalised so that  $X_n^\alpha$  is the fractional amount of the  $\mathcal{O}(g_0^\alpha a^n)$  correction at a standard value of  $g_0 = 1$  (i.e.  $\beta = 6$ ). For  $M_\rho, f_\pi$  and the  $1P-1S$  splitting,  $\alpha = 0$  &  $n = 1$ ; for  $\sigma$  and  $\Delta\beta$ ,  $\alpha = 0$  &  $n = 2$ ; and for  $r_0$ ,  $\alpha = 2$  &  $n = 2$ .

Lattice Monte Carlo data taken from many different collaborations are fit to eq.(2) (see [5] for a list of these references). This provides the values for  $\Lambda$ ,  $X_n^\alpha$ , and the  $\chi^2$  as listed in the table. Also shown in the table are the fits to (2-loop)  $g_0$ -PT. (The fit in this case is eq.(1) with  $g \equiv g_0$ .)

We see that leading order “lattice distorted-PT” fits the data very well compared to the  $g_0$ -PT case, with the  $\chi^2/dof$  down by an order of magnitude or more.

As a further check of the method, we include in the fit the next-to-leading term in  $a$ . However, due to the large statistical errors in some of the lattice data we perform this fit only for  $\sigma, r_0$  and  $\Delta\beta$  where the statistical errors are very small. The fitting function appropriate for these

quantities is:

$$a_L^{-1}(g_0^2) = \frac{\Lambda}{f_{PT}(g_0^2)} \times \left[ 1 - X_n^\alpha \frac{g_0^\alpha f_{PT}^n(g_0^2)}{f_{PT}^n(1)} - X_{n+2}^\alpha \frac{g_0^\alpha f_{PT}^{n+2}(g_0^2)}{f_{PT}^{n+2}(1)} \right]. \quad (3)$$

The results of these fits are displayed in the table.

Obviously in the limit of infinite statistical precision, adequate fits to the lattice distorted-PT formula would only be obtained if the  $\mathcal{O}(a^n)$  terms were included to all orders. The fact that it is necessary to go to next-to-leading order for the  $\sigma, r_0$  and  $\Delta\beta$  data to obtain a sensible  $\chi^2/dof$  simply states that these data have sufficiently small statistical errors to warrant this order fit.

The rest of this section comments on the results of these fits.

It is clear that for  $\sigma, r_0$  and  $\Delta\beta$  data, the agreement between the data and lattice distorted-PT is remarkable considering the tiny statistical errors in the lattice data.

Another important finding is that the values of  $\Lambda$  for the various quantities are all consistent with  $\Lambda = 2.15$  MeV within around  $1\sigma$  with the only exception being the string tension. This slight discrepancy can easily be explained by the effects of quenching, and the uncertainties in the experimental value of  $\sigma$ . Taking  $\Lambda = 2.15 \pm 10\%$  MeV as an overall average, and converting to the

$\overline{MS}$  scheme, we have  $\Lambda_{\overline{MS}}^{(N_f=0)} = 190 \pm 20 MeV$ . This compares well with other lattice determinations and therefore supports the validity of this approach.

The typical values of  $X_n^\alpha$  in the table are 20-40%. A study at  $\beta = 6.4$  [6] found that non-perturbative determinations of the renormalisation constant of the local vector current vary by 10-20% depending on the matrix element used. Since the spread in  $Z_V^{Ren}$  has been interpreted as  $\mathcal{O}(a)$  effects [7], we can assume that  $\mathcal{O}(a)$  effects of around 20-40% are reasonable at  $\beta = 6.0$ .

The coefficients for the second order terms are an order of magnitude less than the first order terms. This follows our expectation that the expansion in  $f_{PT}$  in eq.(3) forms a convergent series.

One of the most exciting features of the lattice distorted-PT approach is that it can reproduce the behaviour of  $\Delta\beta$ . The interpretation of the well-known discrepancy between  $g_0$ -PT and Monte Carlo  $\Delta\beta$  data has been problematic in the past. For example, in [8], a fit was attempted to their  $\Delta\beta$  data using a coupling with two free parameters. A good fit was obtained only for an unphysical value of one of these parameters, leaving the explanation of the discrepancy open. The lattice distorted-PT approach solves this problem.

Finally, as far as the fit to  $M_\rho, f_\pi$  and the  $1P - 1S$  splitting are concerned, the errors in the lattice data are large enough to allow many functional forms. Thus these data do not constrain the lattice distorted-PT fit (or fits from other schemes).

### 3. Fits Using a Renormalised Perturbation Theory

In this section we fit the Monte Carlo data for  $a^{-1}$  to the following functional form:

$$a_L^{-1}(g_0^2) = \frac{\Lambda}{f_{PT}((g')^2)}, \quad (4)$$

where  $g'$  is some “renormalised” coupling which is in turn a function of the bare coupling  $g_0$ . Note that in this philosophy, the failure of asymptotic (i.e. perturbative) scaling is explained by higher order terms in perturbation theory and lattice artifacts are assumed to be negligible.

We studied two definitions of  $g'$ : (i) The  $\overline{MS}$ -like coupling [9],

$$\frac{1}{g_{\overline{MS}}^2(\pi/a)} = \frac{1}{g_0^2} < \frac{1}{3} Tr U_{plaq} >_{MC} + 0.025.$$

and the scheme of [1] based on the plaquette,

$$\frac{1}{g_E^2} = \frac{1/3}{1 - < \frac{1}{3} Tr U_{plaq} >_{MC}}.$$

(Note in the full paper, alternative definitions of  $g'$  are studied as well as those above [5].)

The results of fits using these definitions of  $g'$  in the fitting function Eq.(4) are displayed in the table in the rows headed  $g_{\overline{MS}}$  and  $g_E$ .

As can be seen they do not reproduce the Monte Carlo results nearly as well as the fits from the lattice distorted-PT.

### 4. Conclusions

This talk studies the question of why (dimensionful) lattice Monte Carlo quantities do not follow the predictions of 2-loop perturbation theory in the bare coupling. The conventional answer to this problem is that higher order terms in  $g_0^2$  spoil the behaviour of perturbation theory, and that therefore an improved coupling is required. An alternative approach is presented here where the effects of  $\mathcal{O}(a)$  are shown to be able to provide the mismatch. The quality of the fits using this latter approach, and various arguments outlined in Sec.2 support this philosophy. Further studies are required to unambiguously confirm this issue.

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